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Local Lipschitz Constants

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Let X be a closed subset of $I = [-1, 1]$. For $f \in C[X]$, the local Lipschitz constant is defined to be

$$\lambda_{n\delta}(f) = \sup\{\|B_n(f) - B_n(g)\|/\|f-g\| : 0 < \|f-g\| \leq \delta\},$$

where $B_n(g)$ is the best approximation in the sup norm to g on X from the set of polynomials of degree at most n. It is shown that under certain assumptions the norm of the derivative of the best approximation operator at f is equal to the limit as $\delta \to 0$ of the local Lipschitz constant of f, and an explicit expression is given for this common value. The, possibly very different, characterizations of local and global Lipschitz constants are also considered.

1. INTRODUCTION

Let X be a closed subset of $I = [-1, 1]$, and for any $f \in C[X]$, define $\|f\| = \sup\{|f(x)| : x \in X\}$. Denote the set of all polynomials of degree n or less by π_n, and let the best approximation to f from π_n on X be designated by $B_n(f)$. Define the *global Lipschitz constant* by

$$\lambda_n(f) = \sup\{\|B_n(f) - B_n(g)\|/\|f-g\| : g \neq f\}. \tag{1.1}$$

It is known that $\lambda_n(f)$ is finite for each $f \in C[X]$ [3, p. 82]. A number of interesting papers [1, 4, 5, 7, 8, 10–12, 14] have considered the behavior of $\lambda_n(f)$, depending on f, n, and X.

The strong unicity constant $M_n(f)$ is defined by

$$M_n(f) = [\inf\{(\|f-P\| - \|f-B_n(f)\|)/\|P-B_n(f)\| : P \in \pi_n, P \neq B_n(f)\}]^{-1}, \tag{1.2}$$

and is intimately related to the global Lipschitz constant. However, the behavior of the strong unicity constant has been studied more extensively

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(for example, see [2, 9] and the references of these papers) than the behavior of the global Lipschitz constant, primarily because the strong unicity constant can sometimes be determined by examining the norms of a certain collection of $n+2$ interpolating polynomials [10, Eq. (2.13); 16].

If the cardinality of X (denoted $|X|$) is $n+2$, an explicit relationship between $\lambda_n(f)$ and $M_n(f)$ has been exhibited. In this case, Henry *et al.* [7] have shown that

$$\lambda_n(f) = 2M_n(f)/(M_n(f)+1). \tag{1.3}$$

This equality will be re-examined in Section 3 of the current paper.

Much of the research on the behavior of the strong unicity constant alluded to above has dealt with the asymptotic growth of (1.2) as a function of dimension. For $|X| > n+2$, parallel research on the behavior of the global Lipschitz constant is not nearly as prevalent, primarily because, in contrast to the strong unicity constant, no concise characterizations of the global Lipschitz constant are known. In an early paper on strong unicity and Lipschitz constants, Henry and Roulier [10] do construct an $f \in C[I]$ whose global Lipschitz constant as a function of dimension has asymptotic growth of the order 2^n. Perhaps the most comprehensive paper on the asymptotic growth of the global Lipschitz constant as a function of dimension is by Kroó [12]. A principal result in Kroó's paper states that for any sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$, where $a_n \to +\infty$ as $n \to +\infty$, there exists a function $f \in C[X]$ ($X = [a, b]$) such that

$$\overline{\lim_{n\to\infty}} \frac{[\lambda_n(f)]^{1/n}}{a_n} = +\infty.$$

Let

$$e_n(f)(x) = f(x) - B_n(f)(x). \tag{1.4}$$

Then the extremal set of $e_n(f)$ is defined to be

$$E_n(f) = \{x \in X \colon |e_n(f)(x)| = \|e_n(f)\|\}. \tag{1.5}$$

If $|E_n(f)| = n+2$, then Kroó [13] gives a lower bound for $\lambda_n(f)$. More specifically, this lower bound is given in terms of the "derivative" of the operator of best approximation.

DEFINITION 1 [13]. Let $f, g \in C[X]$. Then

$$D_f B_n(g) = \lim_{t\to 0} \frac{B_n(f+tg) - B_n(f)}{t}, \tag{1.6}$$

if the limit exists. We say $D_f B_n(g)$ is the derivative of $B_n(f)$ in the direction of g.

It was shown in [13] that if $|E_n(f)| = n+2$, and $X = [0, 1]$, then $D_f B_n(g)$ exists for all $g \in C[X]$, and $D_f B_n$ is a linear operator which maps $C[X]$ into π_n. In fact, this result and the next theorem are valid for X closed and contained in I with $|X| \geqslant n+2$.

THEOREM 1 (Kroó [12]). *If $f \in C[I]$ and if $|E_n(f)| = n+2$, then*

$$\lambda_n(f) \geqslant \|D_f B_n\|. \tag{1.7}$$

Theorems 2 and 3 in [7] essentially show that the lower bound in (1.7) is not sharp. However, $\|D_f B_n\|$ is closely related to the *local Lipschitz constant* defined below. In fact, the main objective of the current paper is to show that $\|D_f B_n\|$ is the limit of a certain sequence of local Lipschitz constants.

2. LOCAL LIPSCHITZ CONSTANTS

DEFINITION 2. For fixed $f \in C[X]$ and $\delta > 0$, let

$$\lambda_{n\delta}(f) = \sup\{\|B_n(f) - B_n(g)\| / \|f-g\| : 0 < \|f-g\| \leqslant \delta\}. \tag{2.1}$$

Then $\lambda_{n\delta}(f)$ is the local Lipschitz constant determined by f and δ.

Although most of the research previously referred to has dealt with the global Lipschitz constant, the local Lipschitz constant is of interest for at least two reasons. First, knowledge of the local Lipschitz constant may aid research efforts on global Lipschitz constants. Second, in applications one is often interested in computing over a discretized interval an approximation to a function which is not known exactly (e.g., noisy data); knowing the size of the local Lipschitz constant allows one to determine an upper bound for the norm of the difference between the computed approximation and the approximation actually being sought.

The next theorem is the main theorem of this paper.

THEOREM 2. *Suppose X is a closed subset of I, $|E_n(f)| = n+2$, and $\|e_n(f)\| \neq 0$. Let $X_n = E_n(f) = \{x_0, ..., x_{n+1}\}$. Define $\{q_i\}_{i=0}^{n+1} \subseteq \pi_n$ by*

$$q_i(x_j) = (-1)^j, \tag{2.2}$$

$j=0, 1,\dots, n+1$; $j \neq i$; $i=0, 1,\dots, n+1$. *Then*

$$\lim_{\delta \to 0} \lambda_{n\delta}(f) = \left\| \sum_{i=0}^{n+1} \frac{|q_i|}{1+|q_i(x_i)|} \right\| = \|D_f B_n\|. \tag{2.3}$$

The proof of Theorem 2 will be deferred until after several lemmas have been introduced. It is worth noting that the polynomials defined in (2.2) are the interpolating polynomials mentioned earlier in connection with (1.2); in fact, if $|E_n(f)| = n+2$, then [10]

$$M_n(f) = \max_{0 \leqslant i \leqslant n+1} \|q_i\|. \tag{2.4}$$

3. Lemmas

Lemma 1. *Let $f \in C[X]$, and suppose that $X_n = \{x_0, x_1,\dots, x_{n+1}\} \subseteq X$. Then*

$$B_n(f, X_n) = \sum_{l=0}^{n+1} \frac{(-1)^{l+1} f(x_l)}{1+|q_l(x_l)|} q_l. \tag{3.1}$$

The notation employed in (3.1) is to emphasize that $B_n(f, X_n)$ is the best approximation to f from π_n over X_n instead of X. The proof of (3.1) is given in [6]. We observe (3.1) implies that for any $p \in \pi_n$,

$$p = \sum_{l=0}^{n+1} \frac{(-1)^{l+1} p(x_l)}{1+|q_l(x_l)|} q_l. \tag{3.2}$$

Lemma 2. *For the polynomials defined by* (2.2), *the following properties are valid*:

$$\text{(a)} \quad \sum_{i=0}^{n+1} 1/(1+|q_i(x_i)|) = 1, \tag{3.3}$$

and

$$\text{(b)} \quad \sum_{i=0}^{n+1} q_i/(1+|q_i(x_i)|) \equiv 0. \tag{3.4}$$

Proof. Define f on X_n by $f(x_i) = (-1)^{i+1}$. Then $B_n(f, X_n) \equiv 0$, and (b) follows from Lemma 1. Applying (b) with $x = x_0$, we get

$$0=\sum_{i=0}^{n+1}\frac{q_i(x_0)}{1+|q_i(x_i)|}=\frac{-|q_0(x_0)|}{1+|q_0(x_0)|}+\sum_{i=1}^{n+1}\frac{1}{1+|q_i(x_i)|}$$
$$=-1+\sum_{i=0}^{n+1}\frac{1}{1+|q_i(x_i)|}$$

and (a) holds. ∎

DEFINITION 3. For sets U and V both contained in I, the density of U in V is defined by

$$\rho(U, V)=\sup_{v\in V}\inf_{u\in U}|u-v|. \tag{3.5}$$

A proof of the next lemma appears in [12] for the case $X=[a, b]$, but the same proof works for X (closed) $\subseteq I$.

LEMMA 3. *Let f and $\{f_k\}_{k=1}^{\infty}$ belong to $C[X]$, and assume that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f. Then for fixed n, $\rho(E_n(f), E_n(f_k))\to 0$ as $k\to\infty$.*

The following lemma is crucial to the proof of Theorem 2.

LEMMA 4. *Let $f\in C[X]$ and suppose $\|e_n(f)\|\neq 0$. Suppose that $X_n=E_n(f)=\{x_0,..., x_{n+1}\}$. For $g\in C[X]$, select an alternant $Y_n=\{y_0,..., y_{n+1}\}$ and let $d(g)=\max_{0\leqslant l\leqslant n+1}|x_l-y_l|$. Then $\lim_{\delta\to 0}\sup_{\|f-g\|\leqslant\delta} d(g)=0$.*

Proof. Suppose the conclusion is false. Then there exist a sequence $\{g_k\}\subseteq C[X]$ and an $\varepsilon>0$ such that $g_k\to f$ uniformly on X and for each k there exists an alternant $Y_k=\{y_0^k,..., y_{n+1}^k\}$ such that $d(g_k)\geqslant\varepsilon$. Extracting appropriate subsequences, we may assume that $y_i^k\to y_i$, $i=0,..., n+1$. By Lemma 3, $\{y_0,..., y_{n+1}\}\subseteq\{x_0,..., x_{n+1}\}$. Since f and B_n are continuous, $e_n(f)(y_i)$ alternate in sign, $i=0,..., n+1$, and so $x_i=y_i$, $i=0,..., n+1$. This is a contradiction. ∎

The following notation is employed by Kroó [13]. For $E_n(f)=\{x_0, x_1,..., x_{n+1}\}$, let

$$U_i(x)=\begin{vmatrix}1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1\\ x_0 & x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_n & x\\ \vdots & \vdots & & \vdots & \vdots & & & \vdots\\ x_0^n & x_1^n & \cdots & x_{i-1}^n & x_{i+1}^n & \cdots & x_n^n & x^n\end{vmatrix},$$

$i=0, 1, \ldots, n$, $U_i = U_i(x_{n+1})$, $i=0, 1, \ldots, n$, and

$$U_{n+1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix}.$$

If $g \in C[X]$, define $\bar{Q}(g \mid f) = \sum_{i=0}^{n+1} (-1)^i g(x_i)\, U_i$, and denote the polynomial of degree at most n that interpolates g at $\{x_i\}_{i=0}^{n}$ by $P_f(g)$. Then in [13] Kroó shows that the derivative operator given by (1.6) may be expressed as

$$D_f B_n(g)(x) = P_f(g)(x) - P_f(e_n(f))(x) \frac{\bar{Q}(g \mid f)}{\bar{Q}(f \mid f)}, \qquad x \in X. \tag{3.6}$$

LEMMA 5. *Let f and g be elements of $C[X]$, and suppose that $E_n(f) = \{x_0, x_1, \ldots, x_{n+1}\}$. Let $(D_f B_n)(g)$ be as described in* (1.6). *Then*

$$(D_f B_n)(g)(x) = \sum_{i=0}^{n+1} \frac{(-1)^{i+1} g(x_i)}{1 + |q_i(x_i)|} q_i(x), \tag{3.7}$$

where q_i, $i = 0, 1, \ldots, n+1$ is defined by (2.2).

Proof. Let $l_j^{n+1} \in \pi_n$ be defined by $l_j^{n+1}(x_i) = 0$, $i = 0, 1, \ldots, n$, $i \neq j$, $l_j^{n+1}(x_j) = 1$, $j = 0, 1, \ldots, n$. Then the $\{l_j^{n+1}\}_{j=0}^{n}$ are the $n+1$ Lagrange interpolating polynomials [15, p. 87] determined by $E_n(f) - \{x_{n+1}\}$. We first claim that

$$(D_f B_n)(g)(x) = P_f(g)(x) - q_{n+1}(x) \sum_{i=0}^{n+1} \frac{(-1)^i g(x_i)}{1 + |q_i(x_i)|}, \tag{3.8}$$

where $P_f(g)(x) = \sum_{i=0}^{n} g(x_i)\, l_i^{n+1}(x)$. In fact, Eq. (3.8) may be established by noting for $p \in \pi_n$ that $\bar{Q}(p \mid f) = 0$, and that $1/(1 + |q_i(x_i)|) = U_i / \sum_{k=0}^{n+1} U_k$, $i = 0, 1, \ldots, n+1$. Utilizing these observations in the right-hand side of (3.6) leads directly to (3.8). An elementary calculation now establishes that the right-hand sides of (3.7) and (3.8) are equal at x_j, $j = 0, 1, \ldots, n$, which in turn implies (3.7). ∎

Lemmas 1–5 will be used in the next section to prove Theorem 2.

4. Theorem and Corollaries

Proof of Theorem 2. Select $\varepsilon > 0$ where $\varepsilon < \frac{1}{3} \min_{1 \leqslant l \leqslant n+1}(x_l - x_{l-1})$ and, for any $l = 0, ..., n+1$ and any $x \in [x_l - \varepsilon, x_l + \varepsilon] \cap X$,

$$\operatorname{sgn} e_n(f)(x) = \operatorname{sgn} e_n(f)(x_l) \tag{4.1}$$

and

$$|e_n(f)(x)| > \frac{1}{2} \|e_n(f)\|. \tag{4.2}$$

By Lemma 4 and the continuity of f and B_n, choose $\delta > 0$ so that $0 < \|f - g\| \leqslant \delta$ implies that

$$d(g) < \varepsilon \tag{4.3}$$

and

$$\|e_n(f) - e_n(g)\| < \frac{1}{2} \|e_n(f)\|. \tag{4.4}$$

For any $l = 0, ..., n+1$ and any $x \in [x_l - \varepsilon, x_l + \varepsilon] \cap X$, (4.1), (4.2), and (4.4) imply that

$$\operatorname{sgn} e_n(g)(x) = \operatorname{sgn} e_n(f)(x_l). \tag{4.5}$$

For this choice of δ, let $X_{n\delta} = \{x_{0\delta}, x_{1\delta}, ..., x_{n+1,\delta}\}$ be an alternant for $e_n(g)$. Then from (3.1) and (3.2), we see that

$$\begin{aligned} B_n(f) - B_n(g) = &\sum_{l=0}^{n+1} (-1)^{l+1} [f(x_l) - B_n(g)(x_l) - e_n(g)(x_{l\delta})] \frac{q_l}{1 + |q_l(x_l)|} \\ &+ \sum_{l=0}^{n+1} \frac{(-1)^{l+1} e_n(g)(x_{l\delta})}{1 + |q_l(x_l)|} q_l. \end{aligned} \tag{4.6}$$

Since $e_n(g)(x_{l\delta}) = (-1)^l \|e_n(g)\| \operatorname{sgn} e_n(g)(x_{0\delta})$, (3.4) of Lemma 2 implies that the last term in (4.6) is zero. Let

$$h_{l\delta} = f(x_l) - g(x_{l\delta}) + B_n(g)(x_{l\delta}) - B_n(g)(x_l). \tag{4.7}$$

Then (4.6) may be rewritten as

$$B_n(f) - B_n(g) = \sum_{l=0}^{n+1} \frac{(-1)^{l+1} h_{l\delta}}{1 + |q_l(x_l)|} q_l. \tag{4.8}$$

We now claim that there exists a function $R(l, g)$ such that

$$(f - g)(x_{l\delta}) + R(l, g) \leqslant h_{l\delta} \leqslant (f - g)(x_l) \tag{4.9}$$

if $e_n(f)(x_l) > 0$, and such that

$$(f-g)(x_l) \leqslant h_{l\delta} \leqslant (f-g)(x_{l\delta}) + R(l, g) \tag{4.10}$$

if $e_n(f)(x_l) < 0$, where for both (4.9) and (4.10),

$$|R(l, g)| \leqslant K\varepsilon \, \|f-g\|. \tag{4.11}$$

Here K is a positive constant that does not depend on g or δ.

We examine (4.9) and (4.11). Since $e_n(g)$ alternates on $X_{n\delta}$, (4.5) and Lemma 4 imply that $e_n(g)(x_{l\delta})$ and $e_n(f)(x_l)$ have the same sign; consequently, $e_n(g)(x_{l\delta}) \geqslant e_n(g)(x_l)$. Therefore

$$h_{l\delta} = f(x_l) - g(x_l) + e_n(g)(x_l) - e_n(g)(x_{l\delta}) \leqslant (f-g)(x_l). \tag{4.12}$$

To establish the lower bound in (4.9), we observe that

$$\begin{aligned} h_{l\delta} &= (f-g)(x_{l\delta}) + e_n(f)(x_l) - e_n(f)(x_{l\delta}) + (B_n(f) - B_n(g))(x_l) \\ &\quad - (B_n(f) - B_n(g))(x_{l\delta}) \\ &\geqslant (f-g)(x_{l\delta}) + (B_n(f) - B_n(g))(x_l) - (B_n(f) - B_n(g))(x_{l\delta}) \\ &= (f-g)(x_{l\delta}) + (B_n(f) - B_n(g))'(\xi(l, g))(x_l - x_{l\delta}) \\ &= (f-g)(x_{l\delta}) + R(l, g), \end{aligned} \tag{4.13}$$

where

$$R(l, g) = (B_n(f) - B_n(g))'(\xi(l, g))(x_l - x_{l\delta}). \tag{4.14}$$

Applying Markoff's Inequality [3, p. 91] and Lemma 4 to (4.14) results in

$$|R(l, g)| \leqslant \varepsilon n^2 \, \|B_n(f) - B_n(g)\|_I. \tag{4.15}$$

The subscript I in (4.15) is to indicate that the norm is over I, even though $B_n(f)$ and $B_n(g)$ are best approximations to f and g, respectively, over X. Now writing $(B_n(f) - B_n(g))(x) = \sum_{i=0}^{n} b_i x^i$, we have $\|B_n(f) - B_n(g)\|_I \leqslant (n+1) \max\{|b_i| : 0 \leqslant i \leqslant n\}$. By a standard argument using the linear independence of $\{1, x, ..., x^n\}$ on X [15, pp. 1–3], there exists a constant $C_1(n)$ depending only on n and X such that $\max\{|b_i| : 0 \leqslant i \leqslant n\} \leqslant C_1(n) \, \|B_n(f) - B_n(g)\|$. Thus inequality (4.15) implies that

$$|R(l, g)| \leqslant \varepsilon n^2 (n+1) \, C_1(n) \, \|B_n(f) - B_n(g)\|. \tag{4.16}$$

Applying (1.1) to (4.16) yields $|R(l, g)| \leqslant \varepsilon n^2 (n+1) \, C_1(n) \, \lambda_n(f) \, \|f-g\|$. This inequality, (4.13), and (4.12) combine to establish (4.9) and (4.11).

Since the proof of (4.10) and (4.11) is similar to the just completed proof of (4.9) and (4.11), we omit those details.

From (4.8), (4.11), and either (4.9) or (4.10), we see that

$$\frac{\|B_n(f)-B_n(g)\|}{\|f-g\|} \leqslant \left\| \sum_{l=0}^{n+1} \frac{|q_l|}{1+|q_l(x_l)|} \right\| (1+K\varepsilon) = \Phi(X, X_n)(1+K\varepsilon),$$

where

$$\Phi(X, X_n) = \left\| \sum_{l=0}^{n+1} \frac{|q_l|}{1+|q_l(x_l)|} \right\|. \tag{4.17}$$

Therefore (2.1) implies that

$$\lambda_{n\delta}(f) \leqslant \Phi(X, X_n)(1+K\varepsilon). \tag{4.18}$$

Since ε can be chosen arbitrarily small if we choose $\delta > 0$ sufficiently small, we have

$$\lim_{\delta \to 0} \lambda_{n\delta}(f) \leqslant \Phi(X, X_n). \tag{4.19}$$

From Lemma 5, we can infer that

$$\|D_f B_n\| = \Phi(X, X_n). \tag{4.20}$$

But a direct argument as in [12] shows that $\lambda_{n\delta}(f) \geqslant \|D_f B_n\|$ for any $\delta > 0$, so we have

$$\lim_{\delta \to 0} \lambda_{n\delta}(f) \geqslant \Phi(X, X_n). \tag{4.21}$$

Thus combining (4.17), (4.19), (4.20), and (4.21) yields (2.3), and the proof of Theorem 2 is complete. ∎

Theorem 2 extends results appearing in [6] to the more difficult case when the cardinality of X may be infinite.

COROLLARY. *If* $|X| = n+2$, *then the local and global Lipschitz constants both satisfy* (1.3) *and are independent of* f.

Although this corollary is proven in [7] by different methods, it is instructive to see how the result follows from Theorem 2.

Proof of the Corollary. First, we note that $X_n = X$. From (3.1) of Lemma 1, we have that

$$\frac{\|B_n(f)-B_n(g)\|}{\|f-g\|} \leqslant \Phi(X_n, X_n)$$

for all $g \in C[X_n]$.

Thus from (1.1), $\lambda_n(f) \leqslant \Phi(X_n, X_n)$. Since $\lambda_n(f) \geqslant \lambda_{n\delta}(f)$ for all $f \in C[X_n]$, Theorem 2 implies that

$$\lambda_n(f) = \Phi(X_n, X_n).$$

This equality clearly implies that λ_n is independent of f for $|X| = n+2$. To conclude the proof of this corollary, let k be chosen so that

$$\lambda_n = \sum_{i=0}^{n+1} \frac{|q_i(x_k)|}{1 + |q_i(x_i)|}. \tag{4.22}$$

Then

$$\lambda_n = \sum_{\substack{i=0 \\ i \neq k}}^{n+1} \frac{1}{1 + |q_i(x_i)|} + \frac{|q_k(x_k)|}{1 + |q_k(x_k)|} = \sum_{i=0}^{n+1} \frac{1}{1 + |q_i(x_i)|} + \frac{|q_k(x_k)| - 1}{|q_k(x_k)| + 1}.$$

Now (3.3) of Lemma 2 implies that

$$\lambda_n = 1 + \frac{|q_k(x_k)| - 1}{|q_k(x_k)| + 1} = \frac{2\,|q_k(x_k)|}{|q_k(x_k)| + 1}.$$

This equality shows that if the right side of (4.22) is to be maximal over X_n, then we must have $|q_k(x_k)| = \max_{0 \leqslant l \leqslant n+1} |q_l(x_l)|$. The conclusion follows from (2.4). ∎

5. Observations and Conclusions

Under the assumptions of Theorem 2, we have shown that the norm of the derivative of the best approximation operator is equal to the zero limit of local Lipschitz constants of f. Additionally, an explicit expression for the zero limit of local Lipschitz constants of f involving the interpolating polynomials $\{q_i\}_{i=0}^{n+1}$ defined by (2.2) is given in Theorem 2. Thus the interpolating polynomials used to determine the strong unicity constant when $|E_n(f)| = n+2$ are precisely the ones employed to determine $\lim_{\delta \to 0} \lambda_{n\delta}(f)$.

Based on these observations, it seems reasonable to speculate that an explicit expression for the global Lipschitz constant ($|X| > n+2$) will also involve $\{q_i\}_{i=0}^{n+1}$.

If $|X| = n+2$, the Corollary to Theorem 2 asserts that global and local Lipschitz constants are always equal.

Examples of functions $f \in C[X]$ can be constructed to show that this phenomenon may remain true even if $|X| > n+2$. However, the construction of such examples has proved to be a tedious task. Furthermore, Theorem 3 in [6] and Example 2 in [7] suggest that equal local and global Lipschitz constants may be the exception rather than the rule. It is to be hoped that additional research on the behavior of local and global Lipschitz constants will reveal more explicit connections between local and global Lipschitz constants, Lebesgue constants, and strong unicity constants.

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